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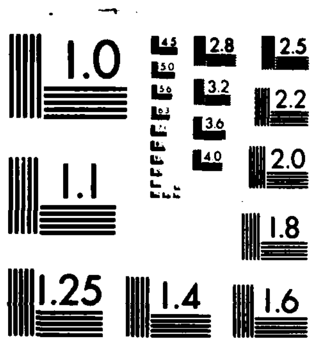

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STABLE VISCOSITIES AND SHOCK PROFILES FOR SYSTEMS OF CONSERVATION LAWS

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ABSTRACT

Wide classes of high order "viscosity" terms are determined, for which small amplitude shock wave solutions of a nonlinear hyperbolic system of conservation laws $u_t + f(u)_x = 0$ are realized as limits of traveling wave solutions of a dissipative system $u_t + f(u)_x = \nu(D_1 u_x)_x + \cdots + \nu^n(D_n u^{(n)})_x$. The set of such "admissible" viscosities includes those for which the dissipative system satisfies a linearized stability condition previously investigated in the case $n = 1$ by A. Majda and this author.

When $n = 1$, we also establish admissibility criteria for singular viscosity matrices $D_1(u)$, and apply our results to the compressible Navier-Stokes equations with viscosity and heat conduction, determining minimal conditions on the equation of state which ensure the existence of the "shock layer" for weak shocks.

AMS (MOS) Subject Classification: 35L65, 35L67, 35B99, 35Q10, 76N10, 35K65.

Key Words: Shock profiles, viscosity, traveling waves, center manifold, compressible Navier-Stokes equations, shock layer.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Many equations of mathematical physics take the form of nonlinear hyperbolic systems of conservation laws. With small dissipative effects neglected, typically smooth solutions must develop discontinuities (shocks) in finite time. Re-incorporating dissipation helps select those discontinuities which are physically relevant. For this purpose, many different sorts of dissipation will do; in particular, the physical viscosity is typically degenerate and not convenient.

In this paper ~~we~~ the authors provide an understanding of what high order viscosity terms smooth the physical discontinuities. A natural class of "admissible" viscosity terms is determined based on a simple linearized stability criterion. In addition, ~~we~~ They determine a class of degenerate second order viscosity terms of physical type which are admissible. These results are applied to the equations of compressible fluid dynamics, to determine what conditions ensure the existence of the "shock layer" with viscosity and heat conduction. This should be of interest to others interested in general equations of state for compressible fluids, such as those investigating phase transitions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STABLE VISCOSITIES AND SHOCK PROFILES
FOR SYSTEMS OF CONSERVATION LAWS

Robert L. Pego

§1. Introduction

Consider a hyperbolic system of m conservation laws in one space dimension,

$$u_t + f(u)_x = 0 \quad , \quad u \in \mathbb{R}^m \tag{1.1}$$

The problem we consider is to determine those matrix n-tuples $(D_1,\dots,D_n)$ with the following property: A simple discontinuous solution of (1.1) in the form

$$u(x,t) = \begin{matrix} u_L & x < st \\ u_R & x > st \end{matrix} \tag{1.2}$$

is the limit of smooth traveling wave solutions $u^\nu = U(\frac{x-st}{\nu})$ of an "approximating" system of higher order,

$$u_t + f(u)_x = \nu(D_1 u_x)_x + \dots + \nu^n(D_n v^{(n)})_x \tag{1.3}$$

as $\nu \downarrow 0$ if (and only if) the solution (1.2) satisfies a suitable entropy condition (see Section 3). Such an n-tuple is called admissible. A solution (1.2) is called a shock wave if the entropy condition is satisfied. (1.2) is a weak solution of (1.1) precisely when the Rankine-Hugoniot jump conditions are satisfied:

$$f(u_R) - f(u_L) - s(u_R - u_L) = 0. \tag{1.4}$$

The traveling wave solution $U(\frac{x-st}{\nu})$ of (1.3) is called a shock profile. With $\xi = (x-st)/\nu$, $U(\xi)$ is required to satisfy the system of ODEs

$$f(U) - f(u_L) - s(U - u_L) = D_1 \frac{dU}{d\xi} + \dots + D_n \frac{d^nU}{d\xi^n} \tag{1.5}$$

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together with boundary conditions

$$U(\xi) \to \begin{matrix} u_L & \text{as} & \xi \to -\infty \\ u_R & \text{as} & \xi \to \infty \end{matrix}$$

(1.6)

$$\frac{d^j U}{d\xi^j} \to 0 \quad \text{as} \quad \xi \to \pm\infty \quad , \quad j = 1,\ldots, n-1.$$

D_j may be a smooth function $D_j\,(u, u_x, \ldots u^{(n-1)})$ for $j = 1,\ldots, n$.

A system (1.3) may naturally be associated with (1.1) in several ways. Physically, (1.1) often arises as a model for a system with small, high order viscosity and/or dispersion terms. Prototype examples are the compressible Navier-Stokes equations in one space dimension, and the KdV-Burgers equation $u_t = (u^2/2)_x + \nu\, u_{xx} + \mu\, u_{xxx}$. Secondly, weak solutions to the Cauchy problem for (1.1) are not unique, and one hopes to identify unique solutions mathematically as limits of solutions of some regularized equation. High order terms may be associated with (1.1) in a third way: given a finite difference approximation to (1.1), it often approximates to better accuracy solutions of an equation with additional dissipative and dispersive terms [5].

We assume that the system (1.1) is strictly hyperbolic, so that if $A(u) = \partial f/\partial u$ is the Jacobian matrix, $A(v)$ has m distinct real eigenvalues, ordered $\lambda_1(u) < \ldots < \lambda_m(u)$ with corresponding right and left eigenvectors $r_k(u)$ and $L_k(u)$, $k = 1,\ldots,m$, satisfying

$$(A - \lambda_j) r_j(u) = 0 \qquad (A^* - \lambda_k)\,\ell_k(u) = 0$$
$$\ell_k \cdot r_j = \delta_{kj}$$

An eigenvalue $\lambda_j(u)$ is called genuinely nonlinear (resp. linearly degenerate) if $\nabla\lambda_j \cdot r_j(u)$ does not vanish (resp. vanishes identically).

The problem above originated with Gelfand [2], who suggested that the entropy condition singles out those simple discontinuities (1.2) which are limits of traveling wave solutions of parabolic systems associated with (1.1) (the case $n = 1$ here). In more concrete form, the investigation of the "shock layer" in gas dynamics dates back much further (see [3]). Most previous work on the problem has been for the case $n = 1$; however, Shapiro [9] and Smoller and Shapiro [10] have obtained some results in the case $n = 2$, assuming genuine nonlinearity.

The present work is based on the analysis (for the case $n = 1$) of Majda and Pego [6], who describe a natural algebraic condition on the viscosity matrix $D = D_1$ called strict stability, and show that any strictly stable matrix is admissible for all weak shocks ($|u_L - u_R|$ small). They also obtain conditions which characterize (up to a degenerate class) those matrices admissible for weak k-shocks (those associated with a particular λ_k, see § 3).

This paper extends the analysis of [6] in two directions. First, admissibility criteria and a notion of strict stability are developed for n-tuples $(D_1, \ldots, D_n)$ for any n (sections 2 and 3). Second, admissibility criteria are established in the case $n = 1$ for <u>singular</u> viscosity matrices $D(u)$, typical in physical systems (section 4). Indeed, in the last section we apply our results to the compressible Navier-Stokes equations of gas dynamics, determining <u>minimal</u> conditions the equation of state must satisfy to ensure the existence of the shock layer for weak shocks, and to ensure that the stability condition holds.

§2 Stable viscosities for strictly hyperbolic systems.

Following [6], the notion of stability for an n-tuple $(D_1, \ldots, D_n)$ may be motivated as follows: Linearize (1.3) at a constant state u_o, obtaining

$$u_t + A(u_o)u_x = \nu D_1 u_{xx} + \ldots + \nu^n D_n u^{(n+1)} \tag{2.1}$$

If (1.3) is to be a good approximation to (1.1), a reasonable requirement to be imposed is that the Cauchy problem for (2.1) be strongly well posed, <u>independent of ν as $\nu \downarrow 0$</u>. In L^2, using the Fourier transform gives this notion an algebraic interpretation:

<u>Definition</u>. We call the n-tuple $(D_1, \ldots, D_n)$ <u>stable</u> for (4.1) if for each $T > 0$ there exists $C(T)$ such that

$$\sup_{\substack{0<t<T \\ \nu>0 \\ \xi \text{ real}}} \| \exp t(-i\xi A(u_o) - \nu\xi^2 D_1 + \ldots + \nu^n (i\xi)^{n+1} D_n) \| < C(T) \tag{2.2}$$

We denote by $S_n(u_o)$ the set of stable n-tuples, considered topologically as a subset of $\mathbb{R}^{nm^2}$. An n-tuple in the <u>interior</u> of $S_n(u_o)$ is called <u>strictly stable</u> at u_o.

Remark. Introduce the matrix polynomial

$$P(\xi) = -i\xi A(u_o) - \xi^2 D_1 + \dots + (i\xi)^{n+1} D_n$$

The condition (2.2) is equivalent to the condition

$$(2.3) \qquad \sup_{\substack{t>0 \\ \zeta \text{ real}}} \| \exp tP(\zeta) \| < C$$

This section is devoted to describing the structure of the set of stable n-tuples. However, we point out that a major objective of this paper is to prove the following:

Theorem 2.1 Suppose the n-tuple $(D_1,\dots,D_n)$ is strictly stable at u_o. If n is even, also assume D_n is nonsingular. Then $(D_1,\dots,D_n)$ is admissible for all shocks in some fixed neighborhood N of u_o. That is, if u_L and u_R are in N and satisfy (1.4), then a corresponding shock profile solution of (1.5), lying in N and satisfying (1.6), exists if and only if Liu's strict entropy condition (E) (see §3) is satisfied by the jump (1.2).

This theorem is a corollary of Theorems 2.3 and 3.1 below. We state here another corollary of Theorem 2.3, giving a convenient sufficient condition for strict stability:

Corollary 2.2. An n-tuple $(D_1,\dots,D_n)$ is strictly stable at u_o if there is a positive definite symmetric matrix E such that $EA(u_o)$ is symmetric and

i) ED_j is symmetric if j is even

ii) $ED_j i^{j-1}$ is positive definite if j is odd

If n is even, we also require that D_n has distinct eigenvalues.

The basic result of this section is Theorem 2.3 below, which characterizes strictly stable n-tuples. Theorem 2.4 completes the description of the set of stable n-tuples for odd n. The difficulties encountered in trying to extend the result to even n are analogous to those involved when (1.1) is not strictly hyperbolic.

Theorem 2.3 The n-tuple $(D_1,\dots D_n)$ is strictly stable at u_o if and only if the following conditions hold:

i) $\ell_k D_1 r_k(u_o) > 0$, $k = 1,\dots,m$.

ii) If $\xi \neq 0$, then $P(\xi)$ has no purely imaginary eigenvalue.

iii) a) If n is odd, the eigenvalues of $B_n i^{n-1}$ have positive real part.

b) If n is even, the eigenvalues of B_n are real and distinct, and if ℓ_k^∞ and r_k^∞ denote corresponding left and right eigenvectors (with $\ell_i^\infty \cdot r_j^\infty = \delta_{ij}$) then $\ell_k^\infty B_{n-1} r_k^\infty i^{n-2} > 0$.

Theorem 2.4 The set $S_n(u_o)$ of stable n-tuples is the closure of its interior if n is odd.

In the rest of this section, we prove 2.2-2.4. We begin by developing necessary criteria for stability. If (2.3) holds, then the eigenvalues of $P(\xi)$ must have nonpositive real part for all real ξ. Using this principle, we may establish:

Proposition 2.5 Assume $(D_1,\ldots,D_n)$ is stable at u_o. Then

i) $\ell_k D_1 r_k(u_o) > 0$, $k = 1,\ldots,m$

ii) For any eigenvalue $\kappa_j(\xi)$ of $P(\xi)$, $\operatorname{Re} \kappa_j(\xi) \leq 0$, $j = 1,\ldots,m$.

iii) a) If n is odd, the eigenvalues of $D_n i^{n-1}$ have nonnegative real part.

b) If n is even, the eigenvalues of D_n are real, and if they are distinct, then $\ell_k^\infty D_n r_k^\infty i^{n-2} > 0$, $k = 1,\ldots,m$ (notation as in 2.3 iii b).

Proof. ii) is immediate. For convenience, we define

$$\begin{aligned} B(\theta) &= -P(\tan\theta)(\cos\theta)^n/\tan\theta \\ &= (\cos\theta)^n i\, A(u_o) + \sin\theta(\cos\theta)^{n-1} D_1 + \ldots + (\sin\theta)^n i^{n-1} D_n \end{aligned}$$

From ii) and continuity, the eigenvalues $\mu_j(\theta)$ of $B(\theta)$ satisfy

$$(\sin\theta)\operatorname{Re}\,\mu_j(\theta) \geq 0\ ,\ -\pi/2 \leq \theta \leq \pi/2\ ,\ j = 1,\ldots,m.$$

Setting $\theta = \pm\,\pi/2$ we obtain iii a) and part of iii b). For i), $B(0) = iA$ has distinct eigenvalues, so for small θ there exist smooth eigenvalues $\mu_k(\theta)$ and eigenvectors $R_k(\theta)$, with $\mu_k(0) = i\lambda_k(u_o)$, $R_k(0) = r_k(0)$, satisfying

$$(B(\theta) - \mu_k(\theta))R_k(\theta) = 0\ .$$

Differentiate, set $\theta = 0$ and dot with $\ell_k(u_o)$, obtaining

$$\ell_k D_1 r_k(u_o) = \mu_k'(0)$$

whence i). For iii b), a similar procedure played at $\theta = \pi/2$ yields

$$-i^{n-2}\ell_k^\infty D_{n-1} r_k^\infty = \mu_k'(\pi/2)$$

Proof of 2.3: The necessity of the conditions is easily established for the most part, by considering scalar perturbations of D_1, D_n, D_{n-1} as appropriate. To show that D_n must have distinct eigenvalues when n is even, we remark that a Jordan block for a single multiple eigenvalue may be perturbed in the (1,2) and (2,1) positions so as to give rise to complex eigenvalues.

The sufficiency of the conditions is established as for the case $n = 1$ in [6], using the Kreiss matrix theorem, and the fact that $B(\theta)$ may be smoothly diagonalized for θ near 0 (and for θ near $\pm\pi/2$ if n is even).

Theorem 2.4 follows directly from 2.5 and 2.3. For if n is odd and $(D_1,\ldots,D_n)$ is stable, then it is easy to check that $(D_1+\delta I, D_2,\ldots,D_{n-1}, D_n + i^{n-1}\delta I)$ is strictly stable for any $\delta > 0$.

Proof of Corollary 2.2. Observe that if M is any real symmetric matrix, and z a complex vector, then z^*Mz is real. Also if M is positive definite, but not necessarily symmetric, then $\mathrm{Re}(z^*Mz) > C|z|^2$. Now suppose $(B(\theta) - \mu_j(\theta))z = 0$. Then

$$\mathrm{Re}\ \mu_j(\theta) \cdot z^*Ez = (\cos\theta)^n \sum_{j\ \mathrm{odd}} (\tan\theta)^j\ \mathrm{Re}\ z^*ED_j i^{j-1} z\ .$$

So for $-\pi/2 < \theta < \pi/2$, $\mathrm{Re}\ \mu_j(\theta) \neq 0$, so ii) of 2.3 holds. Also, for $\theta > 0$ small, $\mathrm{Re}\ \mu_j(\theta) > C\theta$, so $\mu_k'(0) = {}_kD_1r_k > 0$, and i) holds. Similarly, if n is even, iii b) holds, for then

$$\mathrm{Re}\ \mu_j(\theta)\ z^*Ez = (\sin\theta)^n \sum_{j\ \mathrm{odd}} (\cot\theta)^{n-j}\ \mathrm{Re}\ z^*ED_j i^{j-1} z$$

$$> C(\pi/2-\theta) \quad \text{for} \quad \pi/2 - \theta > 0 \quad \text{small.}$$

§ 3 Admissibility for weak k-shocks.

In this section we characterize, up to a degenerate class, those n-tuples $(D_1,\ldots,D_n)$ which are admissible for weak shocks of a particular family. As in [6], the center manifold theorem is used to find a trajectory connecting critical points in an appropriate system of ODEs.

We begin by defining Liu's strict entropy condition. First consider the structure of the Hugoniot set of solutions of the Rankine-Hugoniot conditions (1.4). Fixing u_L, the local structure of this set is well known [1]. In some neighborhood of u_L, the possible solutions u_R lie on m curves, $u_R = \bar{u}^k(\rho)$, $k = 1,\ldots,m$, passing through u_L with corresponding shock speeds $s = s^k(\rho)$, $k = 1,\ldots,m$, satisfying

$$\bar{u}^k(0) = u_L \qquad s^k(0) = \lambda_k(u_L)$$

$$(3.1) \qquad \frac{d\bar{u}^k}{d\rho}(0) = r_k(u_L) \qquad \frac{ds^k}{d\rho}(0) = \tfrac{1}{2}\nabla\lambda_k \cdot r_k(u_L)$$

$$\rho = \ell_k(u_L) \cdot (\bar{u}^k(\rho) - u_L)$$

Liu's strict entropy condition for a k-wave (1.2) with $u_R = \bar{u}^k(\rho_R)$ is that

$s(E) \qquad s^k(\rho) > s = s^k(\rho_R)$ for ρ between 0 and ρ_R.

If $\lambda_k(u)$ is genuinely nonlinear and $|u_L - u_R|$ small, this condition is equivalent to Lax's shock inequalities (see [6]).

Theorem 3.1 Fix $u_o \in \mathbb{R}^m$ and k, $1 \le k \le m$. Assume $\lambda_k(u)$ is not linearly degenerate in any neighborhood of u_o. Assume that the n-tuple $(D_1,\ldots,D_n)$ satisfies the following nondegeneracy conditions at u_o:

i) D_n is nonsingular

ii) $\ell_k D_1 r_k \neq 0$

iii) $-i\xi(\lambda-\lambda_k)(u_o) - \xi^2 D_1 + \ldots + (i\xi)^{n+1} D_n$ is nonsingular for all real $\xi \neq 0$.

Then the following are equivalent:

1) $\ell_k D_1 r_k(u_o) > 0$

2) The n-tuple $(D_1,\ldots,D_n)$ is locally admissible for all k-shocks in a neighborhood of u_o. That is, there exists $\delta > 0$ so that if u_L and u_R in $B_\delta(u_o)$ satisfy the jump conditions (1.4) for some $s = s^k(\rho_R)$, then a shock profile lying in $B_\delta(u_o)$ exists connecting u_L to u_R if and only if Liu's strict entropy condition $s(E)$ is satisfied.

Theorem 2.1 is an immediate corollary of this theorem, using 2.3. We proceed to the proof of 3.1. Our first step is to rewrite (1.5) as an equivalent first order autonomous system of ODEs. Introduce variables $w^j = u^{(j)}$ for $j = 0,1,\ldots,n-1$ and introduce the

parameters $v = u_L$ and s as additional variables. (1.5) is now written, in block form, as

$$
\begin{aligned}
w^o_\xi &= w^1 \\
w^1_\xi &= w^2 \\
&\vdots \\
w^{n-1}_\xi &= D_n^{-1}\Big(f(w^o) - f(v) - s(w^o - v) - \sum_{j=1}^{n-1} D_j w^j\Big) \qquad (3.2)\\
v_\xi &= 0 \\
s_\xi &= 0
\end{aligned}
$$

The existence of a shock profile satisfying (1.6) corresponds to the existence of a trajectory of the system (3.2) connecting the critical point $(u_L, 0\ldots0, u_L, s)$ to the critical point $(u_R, 0\ldots0, u_L, s)$. Our analysis is based on the description of the <u>center manifold</u> of (3.2) at the critical point $(u_o, 0\ldots0, u_o, \lambda_k(u_o))$.

Without loss of generality, assume $u_o = 0$, $\lambda_k(u_o) = 0$. For convenience, introduce the column vector $W = (w^o - v, w^1, \ldots, w^{n-1}, v, s)$, so that $W^o = w^o - v$. Then (3.2) is written

$$W_\xi = T(W) \qquad (3.3)$$

For the statement of the center manifold theorem, consult [6]. To apply the theorem, it suffices to describe two invariant subspaces for the linearization dT at the critical point 0: algebraic eigenspaces corresponding to eigenvalues with zero and nonzero real parts, respectively. To calculate these, compute, in block form on $\mathbb{R}^{nm} \times \mathbb{R}^m \times \mathbb{R}$,

$$dT(0) = \begin{bmatrix} C_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (3.4)$$

where C_o is a block companion matrix,

$$C_o = \begin{bmatrix} 0 & I & 0 & & 0 \\ 0 & 0 & I & & 0 \\ 0 & & & 0 & I \\ D_n^{-1}A & -D_n^{-1}D_1 & & & -D_n^{-1}D_{n-1} \end{bmatrix} w = 0\ .$$

Since $\det D_n \neq 0$ (and $\lambda_k(u_o) = 0$) the characteristic equation for $dT(0)$ may be written

$$\lambda^{m+1} \det(-A+\lambda D_1+ \ldots + \lambda^n D_n) = 0$$

Condition iii) of 3.1 guarantees that $dT(0)$ has no nonzero eigenvalues with zero real parts. Condition ii) of 3.1 guarantees that the zero eigenvalue is semisimple, that is, the algebraic eigenspace for $dT(0)$ for the eigenvalue zero is equal to $\ker dT(0)$. This kernel is spanned by $m + 2$ vectors, $(R_k, 0, 0)$, $(0, 0, 1)$, and $(0,r_j,0)$, $j = 1,\ldots,m$. Here $R_k = (r_k, 0\ldots0) \in \mathbb{R}^{nm}$ and $r_j = r_j(0)$.

Let $Y = \ker dT(0)$ and $X = \text{range } dT(0)$. Then Y and X are complementary invariant subspaces corresponding to eigenvalues with zero and nonzero real parts, respectively. Applying the center manifold theorem, we have (see [6]):

Proposition 3.2 Assume that $(D_1,\ldots, D_n)$ satisfies the nondegeneracy conditions i-iii) at $u_o = 0$ with $\lambda_k(u_o) = 0$. Then there exists $\delta > 0$ and a C^r function $(r\geq2)$ $g : Y \to X$ defined on $B_\delta(0) \cap Y = \{y \in y|\ |y| < \delta\}$ so that

1) $M^* = \{ x + y \in \mathbb{R}^{nm+m+1} \mid x = g(y)\}$ is a locally invariant manifold for the system (3.3).

2) $g(0) = 0$ and $dg(0) = 0$. Thus M^* is tangent to Y at 0.

3) Any trajectory of (3.3) which lies in $B_\delta(0)$ for all ξ lies in M^*. In particular, critical points of (3.3) in $B_\delta(0)$ lie in M^*.

The connection problem for (3.2) is immediately reduced to one for a scalar equation as follows: Define a line in Y parametrized by $y(\eta) = (\eta R_k,u_L,s)$. The curve $W(\eta) = y(\eta) + g(y(\eta))$ lies in M^* while $|y(\eta)| < \delta$, and is itself locally invariant for (3.3), because v and s are constant, while g, mapping into X, is of the form $g(y(\eta)) = (G(\eta,u_L,s),0,0)$, $G \in \mathbb{R}^{nm}$. Returning from W to the (w,v,s) coordinates of (3.2), we find that the system

$$\text{(3.5)}\qquad \begin{aligned} w^o_\zeta &= w^1 \\ w^1_\zeta &= w^2 \\ w^{n-1}_\zeta &= D_n^{-1} \left(f(w^o)-f(u_L)-s(w^o-u_L) - \sum_{j=1}^{n-1} D_j w^j\right) \end{aligned}$$

or $w_\zeta = \bar{T}(w)$ admits an invariant curve

$$w(\eta,u_L,s) = (u_L+\eta r_k,0\ldots0) + G(\eta,u_L,s)$$

so long as $|y(\eta)| = |\eta r_k| + |u_L - u_o| + |s - \lambda_k(u_o)| < \delta$. It follows from part 3) of the above proposition that the point $(u_R, 0\ldots0)$ lies on this invariant curve if $u_R \in B_\delta(u_o)$ and (1.4) holds.

The flow on the invariant curve $w(\eta,u_L,s)$ is now determined by a scalar equation for $\eta(\zeta)$,

(3.6) $$\eta_\xi = F(\eta,u_L,s)$$

where F is determined from the relation

$$w_\eta F(\eta,u_L,s) = \bar{T}(w(\eta,u_L,s))$$

The remainder of the proof, an analysis of the connection problem for the scalar equation (3.6), is virtually identical with that presented in [6] for the case $n = 1$, and is omitted.

§4. Admissibility for singular viscosity matrices

As mentioned in the introduction, viscosity matrices in physical systems are usually singular. In this section we establish quite general admissibility criteria for weak k-shocks for such singular viscosity matrices $D(u)$ (in the case $n=1$). Our result will be applied in the next section in a physical example, the compressible Navier-Stokes equations.

In the case $n = 1$, with $D = D_1(u)$, a shock profile $U(\xi)$ must satisfy the system

(4.1) $$D(U)U_\xi = f(U) - f(u_L) - s(U-u_L)$$

and boundary conditions

$$U(\xi) \to u_L \text{ as } \xi \to -\infty,\ U(\xi) \to u_R \text{ as } \xi \to \infty$$

Theorem 4.1 Fix $u_o \in \mathbb{R}^m$ and k, $1 \leq k \leq m$. Assume $\lambda_k(u)$ is not linearly degenerate in any neighborhood of u_o. Assume the viscosity matrix $D = D_1(u)$ satisfies the following conditions:

i) *$D(u)$ has constant rank in a neighborhood of* u_o

ii) $\ell_k D r_k(u_o) \neq 0$

Since $\det D_n \neq 0$ (and $\lambda_k(u_o) = 0$) the characteristic equation for $dT(0)$ may be written

$$\lambda^{m+1} \det(-A+\lambda D_1 + \dots + \lambda^n D_n) = 0$$

Condition iii) of 3.1 guarantees that $dT(0)$ has no nonzero eigenvalues with zero real parts. Condition ii) of 3.1 guarantees that the zero eigenvalue is <u>semisimple</u>, that is, the algebraic eigenspace for $dT(0)$ for the eigenvalue zero is equal to $\ker dT(0)$. This kernel is spanned by $m + 2$ vectors, $(R_k, 0, 0)$, $(0, 0, 1)$, and $(0, r_j, 0)$, $j = 1,\dots,m$. Here $R_k = (r_k, 0\dots0) \in \mathbb{R}^{nm}$ and $r_j = r_j(0)$.

Let $Y = \ker dT(0)$ and $X = \text{range } dT(0)$. Then Y and X are complementary invariant subspaces corresponding to eigenvalues with zero and nonzero real parts, respectively. Applying the center manifold theorem, we have (see [6]):

<u>Proposition 3.2</u> Assume that $(D_1,\dots, D_n)$ satisfies the nondegeneracy conditions i-iii) at $u_o = 0$ with $\lambda_k(u_o) = 0$. Then there exists $\delta > 0$ and a C^r function $(r \geq 2)$ $g : Y \to X$ defined on $B_\delta(0) \cap Y = \{y \in Y \mid |y| < \delta\}$ so that

1) $M^* = \{ x + y \in \mathbb{R}^{nm+m+1} \mid x = g(y)\}$ is a locally invariant manifold for the system (3.3).

2) $g(0) = 0$ and $dg(0) = 0$. Thus M^* is tangent to Y at 0.

3) Any trajectory of (3.3) which lies in $B_\delta(0)$ for all ξ lies in M^*. In particular, critical points of (3.3) in $B_\delta(0)$ lie in M^*.

The connection problem for (3.2) is immediately reduced to one for a scalar equation as follows: Define a line in Y parametrized by $y(\eta) = (\eta R_k, u_L, s)$. The curve $W(\eta) = y(\eta) + g(y(\eta))$ lies in M^* while $|y(\eta)| < \delta$, and is itself locally invariant for (3.3), because v and s are constant, while g, mapping into X, is of the form $g(y(\eta)) = (G(\eta, u_L, s), 0, 0)$, $G \in \mathbb{R}^{nm}$. Returning from W to the (w,v,s) coordinates of (3.2), we find that the system

$$
(3.5) \quad
\begin{aligned}
w_\zeta^o &= w^1 \\
w_\zeta^1 &= w^2 \\
w_\zeta^{n-1} &= D_n^{-1} \Big(f(w^o) - f(u_L) - s(w^o - u_L) - \sum_{j=1}^{n-1} D_j w^j\Big)
\end{aligned}
$$

or $w_\zeta = \bar{T}(w)$ admits an invariant curve

$$w(\eta,u_L,s) = (u_L+\eta r_k,0\ldots0) + G(\eta,u_L,s)$$

so long as $|y(\eta)| = |\eta r_k| + |u_L - u_o| + |s - \lambda_k(u_o)| < \delta$. It follows from part 3) of the above proposition that the point $(u_R, 0\ldots0)$ lies on this invariant curve if $u_R \in B_\delta(u_o)$ and (1.4) holds.

The flow on the invariant curve $w(\eta,u_L,s)$ is now determined by a scalar equation for $\eta(\zeta)$,

(3.6)
$$\eta_\xi = F(\eta,u_L,s)$$

where F is determined from the relation

$$w_\eta F(\eta,u_L,s) = \bar{T}(w(\eta,u_L,s))$$

The remainder of the proof, an analysis of the connection problem for the scalar equation (3.6), is virtually identical with that presented in [6] for the case $n = 1$, and is omitted.

§4. Admissibility for singular viscosity matrices

As mentioned in the introduction, viscosity matrices in physical systems are usually singular. In this section we establish quite general admissibility criteria for weak k-shocks for such singular viscosity matrices $D(u)$ (in the case $n=1$). Our result will be applied in the next section in a physical example, the compressible Navier-Stokes equations.

In the case $n = 1$, with $D = D_1(u)$, a shock profile $U(\xi)$ must satisfy the system

(4.1)
$$D(U)U_\xi = f(U) - f(u_L) - s(U-u_L)$$

and boundary conditions

$$U(\xi) \to u_L \text{ as } \xi \to -\infty, \; U(\xi) \to u_R \text{ as } \xi \to \infty$$

Theorem 4.1 Fix $u_o \in \mathbb{R}^m$ and k, $1 \le k \le m$. Assume $\lambda_k(u)$ is not linearly degenerate in any neighborhood of u_o. Assume the viscosity matrix $D = D_1(u)$ satisfies the following conditions:

i) $D(u)$ has constant rank in a neighborhood of u_o

ii) $\ell_k D r_k(u_o) \neq 0$

iii) For all real τ, the matrix $[i\tau(A-\lambda_k)-D](u_o)$ is one to one on the subspace $\mathbb{C} \cdot Z_2$, where

$$Z_2 = \{v \in R^m | (A-\lambda_k)(u_o)v \in \text{range } D(u_o)\}$$

Then the following are equivalent:

1) $\ell_k D r_k(u_o) > 0$

2) D is locally admissible for all k-shocks in a neighborhood of u_o. That is, there exists $\delta > 0$ so that if u_L and u_R in $B_\delta(u_o)$ satisfy the jump conditions (1.4) for some $s = s^k(\rho_R)$, then a shock profile lying in $B_\delta(u_o)$ exists connecting u_L to u_R if and only if Liu's strict entropy condition *s(E) is satisfied.*

The main steps in the proof of this theorem are the same as those for Theorem 3.1 above, or Theorem 3.1 of [6]. The difference is that it is a more delicate matter to obtain an autonomous system, like (3.2), to which the center manifold theorem may be applied. Our approach is to use an algebraic condition implied by (4.1) when D is singular to eliminate some variables, then obtain an autonomous system for the remaining variables.

As before, introduce $v = u_L$ and s as variables, writing (4.1) as

$$D(U)\,U_\xi = f(U) - f(v) - s(U - v)$$

$$v_\xi = 0 \qquad (4.2)$$

$$s_\xi = 0$$

We motivate our elimination procedure in the case that D is constant, f linear and $s = \lambda_k(u_o)$. Then (4.2i) is consistent only if $P(A-s)(u-v) = 0$ where P is a projection with $\ker P = \text{range } D$. Write $u = w+v$ for w in $\ker P(A-s)$. In order to reduce (4.2i) to an equation for w_ξ, we should require that $D : \ker P(A-s) \to \text{range } D$ be one-to-one. Note that this entails $Dr_k \neq 0$ and $\ell_k D \neq 0$, for $\dim \ker P(A-s) = \text{rank } D$.

Returning to the case at hand, without loss of generality we assume $u_o = 0$, $\lambda_k(u_o) = 0$. Let $Z_1 = \text{range } D(0)$. Recall that $D(0)$ is one-to-one on Z_2 from iii), so $\dim Z_2 = \dim Z_1 = b$. We may choose (inductively on dimension) a subspace Z_3 complementary to both Z_1 and Z_2, with $\dim Z_3 = m - b$. For u sufficiently small, we

may choose a smooth projection $P(u)$ with range Z_3 and kernel range$(D(u))$; note that $\ker PA(0) = Z_2$.

Given (U,v,s) in $\mathbb{R}^{2m+1}$, write $U - u^3 = u^2 + v$, where u^3 is in Z_3 and u^2 in Z_2. We seek to express u^3 as a function of (u^2,v,s), using the consistency criterion for (4.2i),

(4.3) $\quad P(U)(f(U) - f(v) - s(U - v)) = 0$

We find it convenient to introduce isomorphisms

$$I_2 : \mathbb{R}^b \to Z_2 \qquad I_3 : \mathbb{R}^{m-b} \to Z_3$$

and to write $u^2 = I_2 w$, $u^3 = I_3\tilde{w}$. Then we can apply I_3^{-1} to (4.3), writing

$$h(\tilde{w},w,v,s) = 0$$

where $h : \mathbb{R}^{m-b} \times \mathbb{R}^b \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{m-b}$. In block form, the Jacobian matrix of h at $(\tilde{w},w,v,s) = 0$ is

$$dh(0) = [I_3^{-1}PA(0)I_3,0,0,0]$$

since $\ker PA(0) = Z_2$. The first component is an isomorphism on $\mathbb{R}^b$, since $PA(0)$ is one-to-one on Z_3 to itself. Thus the implicit function theorem applies, so that in a neighborhood of 0 we may write $\tilde{w} = \tilde{w}(w,v,s)$, and indeed the total derivative $d\tilde{w}(0) = [0,0,0]$. We may express

$$U(w,v,s) = I_3\tilde{w}(w,v,s) + I_2w + v$$

and replace (4.2i) by the equation

$$D\,\frac{\partial U}{\partial w}(w,v,s)\;w_\xi = f(U(w,v,s)) - f(v) - s(U(w,v,s) - v)$$

By construction, the right hand side lies in range $(D(U))$. The matrix $D\,\frac{\partial u}{\partial w} : \mathbb{R}^b \to$ range D is an isomorphism at $(w,v,s) = 0$, so also in a neighborhood, since D has constant rank. We may find a smooth generalized inverse

$$(D\,\frac{\partial U}{\partial w})^{\dagger}\,(w,v,s) : \mathbb{R}^m \to \mathbb{R}^b$$

so that $(D\,\frac{\partial U}{\partial w})^{\dagger}(D\,\frac{\partial U}{\partial w}) = I$ in $\mathbb{R}^b$. We have reduced (4.2) to an autonomous system in $\mathbb{R}^b \times \mathbb{R}^m \times \mathbb{R}$,

$$w_\xi = \left(D \frac{\partial U}{\partial w}\right)^\dagger (f(U(w,v,s)) - f(v) - s(U(w,v,s) - v)$$

(4.4) $$v_\xi = 0$$

$$s_\xi = 0$$

which we write $W_\xi = T(W)$ for the variable $W = (w,v,s)$.

We proceed to apply the center manifold theorem to (4.4) at $W = 0$. In block form, since $(\partial U/\partial w)(0) = I_2$ and $(\partial U/\partial v)(0) = I$ in R^m, we have

$$dT(0) = \begin{bmatrix} (DI_2)^\dagger A(0)I_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Condition iii) of the theorem implies that $dT(0)$ has no nonzero imaginary eigenvalues. Condition ii) implies that the eigenvalue 0 is semisimple. The kernel of $dT(0)$ is spanned by $m + 2$ vectors $(0,0,1)$, $(0,r_j,0)$, $j = 1, \ldots, m$, and $(R,0,0)$, where $I_2R = r_k$.

So, defining invariant subspaces $X = \mathit{range}\ dT(0)$, $Y = \ker dT(0)$, we apply the center manifold theorem as in §3 to obtain:

<u>Proposition 4.2</u> Assume that $D(u)$ satisfies the conditions i-iii). Then there exists $\delta > 0$ and a C^r function $(r>2)$ $g : Y \to X$ defined on $B_\delta(0) \cap Y$ in R^{b+m+1} so that

1) $M^* = \{x+y \in R^{b+m+1} | x = g(y)\}$ *is a locally invariant manifold for* (4.4).

2) $g(0) = 0$ and $dg(0) = 0$. Thus M^* is tangent to Y at 0.

3) Any trajectory of (4.4) which lies in $B_\delta(0)$ for all ξ lies in M^*.

As in §3, the connection problem for (4.1) is immediately reduced to one for a scalar equation: Define a line in Y by $y(\eta) = (\eta R, u_L, s)$. *The curve* $W(\eta) = y(\eta) + g(y(\eta))$ is locally invariant for (4.4), meaning it is composed of solution curves. The curve $U(W(\eta))$ is then composed of solution curves of (4.1). We may write this curve

$$U(\eta,u_L,s) = u_L + \eta r_k(0) + G(\eta,u_L,s)$$

where $G = I_3 \tilde{w}\ (\eta R+g(y(\eta)),u_L s) + I_2\ g(y(\eta))$. Note that $G(0) = 0$, $dG(0) = 0$.

If u_R is in $B_\delta(0)$, and satisfies (1.4), then $u_R = U(w_R,u_L,s)$ for some w_R, and (w_R,u_L,s) is a critical point of $T(W)$, so lies in M^*, hence on $W(\eta)$. Thus for some η_R,

$$u_R = u_L + \eta_R r_K + G(\eta_R, u_L, s)$$

The flow on the invariant curve $U(\eta, u_L, s)$ is determined by a scalar equation

$$\eta_\xi = F(\eta, u_L, s)$$

where

$$D(U)\, U_\eta\, F(\eta, u_L, s) = f(U) - f(u_L) - s(U - u_L)$$

The remainder of the proof is identical to that for nonsingular $D(u)$, and may be found in [6].

§5. Weak shock layers in compressible fluid dynamics

Here we use Theorem 4.1 to obtain very weak conditions on the equation of state in the compressible Navier-Stokes equations in one space dimension which guarantee the existence of weak shock profiles. We also make a brief remark concerning the linearized stability of these equations.

In Lagrangian coordinates, the equations are written in conservation form as

$$\begin{aligned} &\tau_t - v_x = 0 \\ (5.1)\qquad &v_t + p_x = \left(\frac{\mu}{\tau} u_x\right)_x \\ &\mathcal{E}_t + (pv)_x = \left(\frac{\mu v}{\tau} v_x\right)_x + \left(\frac{\kappa}{\tau}\theta_x\right)_x \end{aligned}$$

Here x is the Lagrangian mass coordinate, t is time, τ is specific volume, v is velocity, p is pressure, θ is temperature, $\mathcal{E}$ is energy density per unit mass, and μ and κ are, respectively, the coefficients of viscosity and heat conduction. $\mathcal{E} = e + v^2/2$, where e is the internal energy per unit mass. We assume that τ and θ determine the thermodynamic state of the material, and that e and p are given by sufficiently smooth equations of state, $e = e(\tau,\theta)$, $p = p(\tau,\theta)$. μ and κ are positive, and may also depend smoothly on τ and θ. τ, θ, and p are positive.

We assume that the specific heat at constant volume is a positive function:

(5.2) $\qquad c(\tau,\theta) = e_\theta(\tau,\theta) > 0$

So $\theta = \theta(\tau, e)$ and with $u = (\tau, v, \mathcal{E})$, (5.1) may be written in the form

(5.3) $\qquad u_t + f(u)_x = (D(u)u_x)_x$.

We shall see presently that the equation $u_t + f(u)_x = 0$ with diffusion of heat and momentum neglected $(\mu=\kappa=0)$ is strictly hyperbolic if and only if

(5.4) $\quad 0 < -dp/d\tau|_{S \text{ constant}} \equiv \alpha^2$

Here S denotes the entropy, and $S = S(\tau,\theta)$. This function is related to e and p through the Gibbs relations,

(5.5) $\quad \theta dS = de + pd\tau$

Our main result below is that no additional conditions are required to ensure the existence of shock profiles for weak shocks. (The situation is different for stronger shocks; see [8].)

Theorem 5.1 Fix $u_o = (\tau_o, v_o, \mathcal{E}_o) \in \mathbb{R}^3$, $\tau_o > 0$; and assume that (5.2) and (5.4) hold at u_o. Then there exists $\delta > 0$ so that if u_L, u_R and s satisfy the Rankine Hugoniot conditions (1.4) with u_L, u_R in $B_\delta(u_o) = \{u|\ |u-u_o| < \delta\}$, then a shock profile solution $u(x-st)$ of (5.3) lying in $B_\delta(u_o)$ exists connecting u_L to u_R if and only if Liu's strict entropy condition $s(E)$ is satisfied.

The study of the "shock layer" in compressible fluid dynamics has a long history. Most relevant here are the results of Gilbarg [3] and of Liu [4]. Gilbarg established the existence of shock profiles for shocks of any magnitude, under two additional conditions on the equation of state:

(5.6) $\quad 0 < d^2p/d\tau^2|_{S \text{ constant}}$

(5.7) $\quad p_\theta(\tau,\theta) > 0$

The convexity condition (5.6) implies that the eigenvalues $\lambda_1 = -\alpha$ and $\lambda_3 = \alpha$ are genuinely nonlinear. ($\lambda_2 = 0$ is linearly degenerate, see below.) In that case the entropy condition has a simple form. Liu introduced an entropy condition appropriate for the nongenuinely nonlinear case, and showed that, with no heat conduction $(\kappa = 0)$, shock profiles exist for discontinuities satisfying his entropy condition (see §3). This result holds for strong shocks so long as the Hugoniot curves (see (3.1)) remain regular.

Theorem 5.1 is proved by verifying conditions i-iii) and 1) of Theorem 4.1 for the first and third wave fields (k = 1 and 3). Discontinuities associated with the second wave field, called contact discontinuities, cannot satisfy the strict entropy condition $s(E)$.

The Jacobian of $f(u)$ takes the form (with $p(u) = p(\tau,\theta(\tau,\mathcal{E}-v^2/2)))$

$$A(\tau,v,\mathcal{E}) = \begin{bmatrix} 0 & -1 & 0 \\ p_\tau(u) & p_v(u) & p_{\mathcal{E}}(u) \\ vp_\tau(u) & p+vp_v(u) & vp_{\mathcal{E}}(u) \end{bmatrix}$$

The viscosity matrix is

$$\tau D(\tau,v,\mathcal{E}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ -\mu e_\tau(\tau,\theta) & (\mu-\lambda)v & \lambda \end{bmatrix}$$

where $\lambda = \kappa/c > 0$. The information we need will be computed after performing a convenient change of basis (simultaneous similarity transformation of A and D). First, note that

$$p_\tau(u) = p_\tau(\tau,e)\ ,\quad p_{\mathcal{E}}(u) = p_e(\tau,e),\ p_v(u) = -vp_e(\tau,e).$$

With

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e_\tau(\tau,\theta) & v & 1 \end{bmatrix}$$

it follows that $D_1 = T^{-1}\tau D\, T = \mathrm{diag}\,(0,\mu,\lambda)$ and

$$A_1 = T^{-1}A\, T = \begin{bmatrix} 0 & -1 & 0 \\ p_\tau(\tau,e) + p_e e_\tau(\tau,\theta) & 0 & p_e(\tau,e) \\ 0 & p+e_\tau(\tau,\theta) & 0 \end{bmatrix}$$

The eigenvalues of A_1 are $-\alpha$, 0, and α. To see this, compute

(5.8) $$-\alpha^2 = p(\tau,e(\tau,S))_\tau = p_\tau(\tau,e) + p_e(\tau,e)\, e_\tau(\tau,S)$$

and observe that $e_\tau(\tau,S) = -p$ from (5.5).

Condition i) of 4.1 is obviously satisfied. We turn to condition ii). A matrix $R = (r_1, r_2, r_3)$ of right eigenvectors of A_1 is

$$R = \begin{bmatrix} -1 & p_e & -1 \\ -\alpha & 0 & \alpha \\ p+e_\tau & -(p_\tau+p_e e_\tau) & p+e_\tau \end{bmatrix}$$

Thus $A_1 R = R \operatorname{diag}(-a, 0, a)$ and the corresponding matrix of left eigenvectors is $L = R^{-1}$,

$$L = \frac{1}{2a^2}\begin{bmatrix} (p_\tau + p_e e_\tau) & -a & p_e \\ 2(p+e_\tau) & 0 & 2 \\ (p_t + p_e e_\tau) & a & p_e \end{bmatrix} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}$$

We compute

$$\ell_1 D_1 r_1 = \ell_3 D_1 r_3 = (\mu a^2 + \lambda p_e(\tau,e)(p+e_\tau(\tau,\theta))/2a^2$$

From (5.5) and the equality $S_{\tau\theta} = S_{\theta\tau}$ one may verify the thermodynamic identity

(5.9) $\quad \theta p_\theta(\tau,\theta) = p + e_\tau(\tau,\theta)$

Also, $p_\theta(\tau,\theta) = p_e(\tau,e)\, c$, so

$$\ell_1 D_1 r_1 = \ell_3 D_1 r_3 = (\mu a^2 + \lambda\theta c p_e{}^2)/2a^2 > 0$$

and 1) of 4.1 is satisfied. For later reference, we compute

(5.10) $\quad a^2 \ell_2 D_1 r_2 = -\lambda(p_\tau(\tau,e) + p_e(\tau,e)e_\tau(\tau,\theta)) = -\lambda p_\tau(\tau,\theta)$

It remains to verify condition iii) of 4.1 for $k = 1$ and 3. Take $k = 1$.

$$Z_2 = \{ u \in R^3 \mid a u_1 = u_2 \} = \{u \in R^3 \mid (A_1+a)u \in \operatorname{range} D_1\}$$

To show $i\tau(A_1+a) + D_1$ is one-to-one on $\mathbb{C}Z_2$, it suffices to show (cf(5.10)) that

$$\left[i\tau \begin{pmatrix} p_\tau(\tau,\theta)+a^2 & p_e \\ a\theta c p_e & a \end{pmatrix} + \begin{pmatrix} \mu a & 0 \\ 0 & \lambda \end{pmatrix} \right] \begin{pmatrix} 1 \\ z \end{pmatrix} \neq 0$$

for any complex z. A calculation similar to (5.8), using (5.9), yields

(5.11) $\quad p_\tau(\tau,S) = p_\tau(\tau,\theta) - \theta c p_e{}^2(\tau,e)$

Therefore, multiplying by diag $(1, 1/a)$ from the left and diag $(1/\theta c, 1)$ from the right, it suffices to show that $i\tau A_2 + D_2$ is nonsingular for any τ, where

$$A_2 = \begin{pmatrix} p_e{}^2 & p_e \\ p_e & 1 \end{pmatrix} \qquad D_2 = \begin{pmatrix} a\mu/\theta c & 0 \\ 0 & \lambda/a \end{pmatrix}$$

But D_2 is positive definite and A_2 symmetric, so this is true. So iii) of 4.1 holds for $k = 1$. For $k = 3$, replace $-a$ by $+a$ in the argument above. This finishes the proof of Theorem 5.1.

We conclude with a brief remark concerning the linearized stability condition (2.3), where $P(\xi) = -i\xi A - \xi^2 D$. From Proposition 2.5 and (5.10), a necessary condition for (2.3) to hold is that $p_\tau(\tau,\theta) \le 0$. The nondegenerate condition

(5.12) $\quad 0 < -p_\tau(\tau,\theta)$

is stronger than (5.4), by (5.11). *In fact, (5.2) and (5.12) are sufficient* to imply the linearized stability condition, a fact used by Matsumura and Nishida in [7] to establish the full nonlinear stability of the constant state for the compressible Navier Stokes equations in three space dimensions. A proof that the linearized stability condition holds is similar to the proof of Theorem 2.3 for $n = 1$, given in [6].

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Wide classes of high order "viscosity" terms are determined, for which small amplitude shock wave solutions of a nonlinear hyperbolic system of conservation laws $u_t + f(u)_x = 0$ are realized as limits of traveling wave solutions of a dissipative system $u_t + f(u)_x = \nu(D_1 u_x)_x + \ldots$

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ABSTRACT (Continued)

$+ \nu^n (D_n u^{(n)})_x$. The set of such "admissible" viscosities includes those for which the dissipative system satisfies a linearized stability condition previously investigated in the case $n = 1$ by A. Majda and this author.

When $n = 1$, we also establish admissibility criteria for singular viscosity matrices $D_1(u)$, and apply our results to the compressible Navier-Stokes equations with viscosity and heat conduction, determining minimal conditions on the equation of state which ensure the existence of the "shock layer" for weak shocks.